OPTIMAL VARIATION OF POWER FOR THE MOTION OF A BODY OF VARIABLE MASS IN A GRAVITATIONAL FIELD

(OPTIMAL'NOE IZMENENIE MOSHCHNOSTI PRI DVIZHENII TELA PEREMENNOI MASSY V GRAVITATSIONNOM POLE)

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This paper establishes the independence of an optimal power program from control of other guidance parameters. A limiting case of optimal power programming is indicated, and the relations for stepwise reduction of power are given.

1. Generalizing the problem of [1], let us formulate the following variational problem: Let there be given an initial relative composite weight of a power source and fuel, it is required to determine, 1) the law for the power source relative weight reduction and, 2) the law for the change of the thrust acceleration vector ensuring the minimum time of displacement between given locations in a phase space (problem of maximal speed of action).

Let us introduce the following notation: **r** and **v** are the radius vactor and the velocity of a point vector; $\mathbf{R}(\mathbf{r}, t)$ is the acceleration vector due to gravitational forces; **a** is the thrust acceleration vector; G_m, G_N are the weights of fuel and of the power source referenced to starting weights; N is the power of the jet; q is the relative payload.

The above formulated problem is described by the system of equations

$$G_m = -\frac{(G_m + G_N + q)^2}{G_N} a^2 \frac{\alpha}{2g} \qquad (G_N = \alpha N, \quad q = 1 - G_m^{(0)} - G_N^{(0)}) \tag{1.1}$$

$$\mathbf{r} = \mathbf{v}, \qquad \dot{\mathbf{v}} = \mathbf{a} + \mathbf{R}(\mathbf{r}, t)$$

and by the boundary conditions

$$\mathbf{r}_0 = \mathbf{r}(t_0), \quad \mathbf{v}_0 = \mathbf{v}(t_0); \qquad \mathbf{r}_1 = \mathbf{r}(t_1), \quad \mathbf{v}_1 = \mathbf{v}(t_1) \tag{1.2}$$

We select a class of permissible functions for the first part of the formulated problem: We will consider as permissible any piece-wise continuous nonincreasing function

$$\dot{G}_N \leqslant 0 \tag{1.3}$$

For the solution of the problem of maximal speed of action according to the principle of Pontriagin we form the Hamiltonian

$$H = -p_G \frac{(G_m + G_N + q)^2}{G_N} a^2 \frac{\alpha}{2g} + \mathbf{pr} \cdot \mathbf{v} + \mathbf{pv} \cdot (\mathbf{a} + \mathbf{R}) + p_t \tag{1.4}$$

The differential equation for the impulse p_G is of the form

$$\dot{p}_G = p_G 2 \frac{G_m + G_N + q}{G_N} a^2 \frac{\alpha}{2g} \tag{1.5}$$

Function $H(G_N, \ldots)$ can reach its upper bound for one of the functional relationships:

$$G_N = G_m + q \quad \text{or} \quad G_N = \text{const} \tag{1.6}$$

the latter following from (1.3).

In view of the functional relationships (1.6), the expression which is the coefficient of the $a^2\alpha/2g$ term in (1.4) is independent of time, i.e.

$$p_G \frac{(G_m + G_N + q)^2}{G_N} = \text{const} \tag{1.7}$$

This can be verified by differentiation of the latter expression and by taking into account (1.5).

Then, the maximum of the Hamiltonian function H corresponds to the optimal law for the change of the thrust acceleration vector which allows maximal speed of action in displacing between two given locations in the phase space for a given

$$\Phi = \frac{\alpha}{2g} \int_0^T a^2 \, dt \tag{1.8}$$

or, which is equivalent, the maximum of H corresponds to the minimum of the indicated functional φ for a given time of translation [1].

Thus, the formulated variational problem in the beginning of Section 1 is divided into two independent ones:

1) for a given initial relative composite weight of the power source and fuel one finds the law for the reduction of the power source weight

constructed from the extremals of (1.6) which ensures the maximum value* for

$$\Phi = -\int_{G_m^0}^{0} \frac{G_N dG_m}{(G_m + G_N + q)^2} \tag{1.9}$$

2) for a given integral functional Φ one finds the minimal time of translation T between two given points in the space of coordinates and velocities (this part of the problem is referred to the last two equations of the system (1.1)).

A consideration of particular cases of the indicated problem, for example, for a = const [2], leads to the analogous results for the law of power source weight changes.

2. It is required to determine a piece-wise continuous function $G_N = G_N(G_m)$ composed from sections of $G_N = G_m + q$ and G_N = const which ensures a maximum of (1.9) for a given value of q and the following initial condition:

$$G_m^{(0)} + G_N^{(0)} + q = 1 \tag{2.1}$$

Note that first of all, for $q \geqslant 0.5$ the solution of the problem $G_N(G_m)$ cannot contain the straight line $G_N = G_m + q$ since in that case condition (2.1) is meaningless. Consequently, for $q \geqslant 0.5$ the optimal law

$$G_N(G_m) = G_N^0 = \text{const}$$

and

$$\int_0^{G_m^{(0)}} \frac{G_N^{(0)} dG_m}{(G_m + G_N^{(0)} + q)^2} = G_m^{(0)} \left(\frac{1}{G_N^{(0)} + q} - 1\right)$$

From the condition for maximum Φ we have

$$G_N^{(0)} = \sqrt{q} - q \tag{2.2}$$

Then

$$\Phi = (1 - \sqrt{q})^2 \tag{2.3}$$

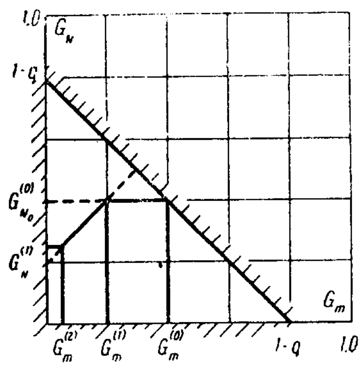

Fig. 1.

* This formulation is equivalent to the following: to find a law for reduction of the power source weight which ensures the minimal composite weight of the power source and fuel for a given value of the integral (1.8).

If $q < 0.5$, then the extremal $G_N(G_m)$ can consist of less than three straight lines (Fig. 1)

$$G_N = G_N^{(0)}, \qquad G_N^{(0)} - q \leqslant G_m \leqslant 1 - q - G_N^{(0)}$$
$$G_N = G_m + q, \qquad G_N^{(1)} - q \leqslant G_m \leqslant G_N^{(0)} - q, \qquad q \leqslant G_N^{(1)} \leqslant G_N^{(0)} \tag{2.4}$$
$$G_N = G_N^{(1)}, \qquad 0 \leqslant G_m \leqslant G_N^{(1)} - q$$

Correspondingly, the integral Φ is of the form

$$\Phi = \frac{G_N^{(1)}}{G_N^{(1)} + q} - \frac{1}{4}\ln G_N^{(1)} + \frac{1}{4}\ln G_N^{(0)} - G_N^{(0)} \tag{2.5}$$

The values of $G_N^{(0)}$ and $G_N^{(1)}$ are found from the condition for maximum of Φ; we have

$$G_N^{(1)} = q, \qquad G_N^{(0)} = 0.25 \tag{2.6}$$

This corresponds to the fact that the broken extremal consists of two straight lines

$$G_N = 0.25 \quad (0.75 - q \geqslant G_m \geqslant 0.25 - q), \qquad G_N = G_m + q \quad (0.25 - q \geqslant G_m \geqslant 0) \tag{2.7}$$

For the indicated values of $G_N^{(1)}$ and $G_N^{(0)}$

$$\Phi = 0.25\left(1 + \ln\frac{1}{4q}\right) \tag{2.8}$$

The solution obtained is valid for the range $0.25 > q > 0$ since then the inequality (2.4) is not violated. We note here, however, that for $1 > q > 0.25$ the extremals will be the straight lines $G_N = \text{const} = \sqrt{q} - q$ (see Formula (2.2)), while the integral $\Phi = (1 - \sqrt{q})^2$ (see Formula (2.3)).

In Fig. 2 the dependence of $G = G_m^{(0)} + G_N^{(0)}$ on Φ is shown by the curve $n = \infty$.

3. In the previous section the optimal dependence of $\Phi(q)$ was obtained which in a certain sense can be called limiting. Below is considered a case of stepwise variation of power along a trajectory and the optimal time and amount of power reduction is established from the condition for the maximum of the function $\Phi(q)$. The integral $\Phi(q)$ is expressed for this case by the sum of n integrals along the sections where power remains constant

$$\Phi(q) = \sum_{i=1}^{n} G_N^{(i)}\left(-\frac{1}{G_N^{(i)} + G_m^{(i+1)} + q} + \frac{1}{G_N^{(i)} + G_m^{(i)} + q}\right) \tag{3.1}$$

The derived expression contains $2n - 1$ unknowns

$$G_N^{(1)}, \ldots, G_N^{(n)}; \qquad G_m^{(2)}, \ldots, G_m^{(n)}$$

Let us find the unknowns $G_N^{(i)}$, $G_m^{(i)}$ from the condition of maximizing $\Phi(q)$ with the additional conditions $G_m^{(n+1)} + G_N^{(n)} + q = 1$, $G_m^{(1)} = 0$. The corresponding system of $2n - 1$ algebraic equations is shown below

$$G_m^{(i)} + q = \sqrt{G_N^{(i)} G_N^{(i-1)}} \quad \text{or} \quad G_N^{(i-1)} = G_N^{(i)} \qquad (i = 2, \ldots, n) \quad (3.2)$$

$$G_N^{(i)} = \sqrt{(G_m^{(i)} + q)(G_m^{(i+1)} + q)} \quad (i = 1, \ldots, n-1)$$

$$G_N^{n} = \sqrt{G_m^{n} + q} - G_m^{n} - q \qquad (3.3)$$

Note that from the condition of maximum Φ with respect to $G_m^{(i)}$ one obtains two noncontradicting groups of Equations (3.2); the second group ($G_N^{(i-1)} = G_N^{(i)}$) shows the possibility of optimal regimes with constant power along the entire trajectory. In conclusion, we will indicate the method of solving the derived system of algebraic equations. The quantities $G_m^{(i)}$ and $G_m^{(i+1)}$ are related by the following recurrence relationship:

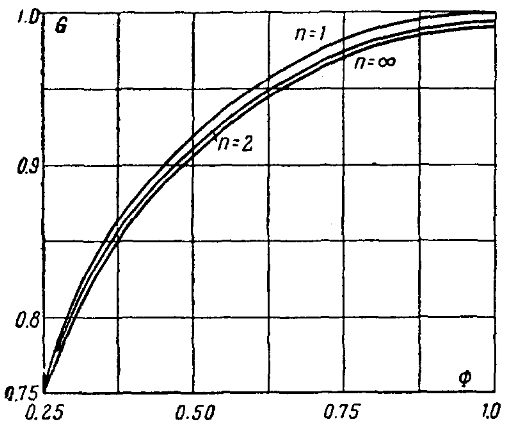

Fig. 2.

$$(G_m^{(i)} + q)^{i+1} = (G_m^{(i+1)} + q)^{i} q \qquad (i = 2, \ldots, n-1) \quad (3.4)$$

This is easily verified by the method of complete induction, while $G_m^{(n)}$ is determined from the equation

$$(G_m^{(n)} + q)^{n+1} = \left(1 - \sqrt{G_m^{(n)} + q}\right)^{n} q \qquad (3.5)$$

Having computed $G_m^{(n)}$ for a given q from (3.5) and having found consequently all $G_m^{(n-1)}, \ldots, G_m^{(2)}$ and $G_N^{(i)}$ from (3.3), it is possible to determine the quantity Φ from (3.1).

Figure 2 shows the examples of $G = G_m^{(0)} + G_N^{(0)}$ dependence on Φ for $n = 1, 2, \infty$.

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